

# EMBEDDING POLYDISK ALGEBRAS INTO THE DISK ALGEBRA AND AN APPLICATION TO STABLE RANKS

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ABSTRACT. It is shown how to embed the polydisk algebras (finite and infinite ones) into the disk algebra  $A(\overline{\mathbb{D}})$ . As a consequence, one obtains uniform closed subalgebras of  $A(\overline{\mathbb{D}})$  which have arbitrarily prescribed stable ranks.

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## INTRODUCTION

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disk,  $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$  its closure, and  $A(\overline{\mathbb{D}})$  the disk-algebra, that is the space of all functions continuous on  $\overline{\mathbb{D}}$  and holomorphic on  $\mathbb{D}$ . In this note I am interested in the question whether there are subalgebras of the disk algebra  $A(\overline{\mathbb{D}})$  that do not have the Bass stable rank one (see below for the definitions). As is well known, Jones, Marshall and Wolff showed that the stable rank of  $A(\overline{\mathbb{D}})$  is one. Whereas in [6] I unveiled for any  $n \in \mathbb{N} \cup \{\infty\}$  subalgebras of  $H^\infty$  on the disk which have stable rank  $n$ , the problem whether these algebras could be chosen to be subalgebras of  $A(\overline{\mathbb{D}})$ , remained open. The examples given in [6] always meet  $H^\infty(\mathbb{D}) \setminus A(\overline{\mathbb{D}})$ . It is a quite recent result developed together with Rudolf Rupp (see Corollary 1.3) that any subalgebra  $B$  of  $A(\overline{\mathbb{D}})$  containing the polynomials and satisfying Royden's property  $(\alpha_0)$  has Bass stable rank one (note that  $B$  is not assumed to be closed in  $A(\overline{\mathbb{D}})$ ). On the other hand, it is easy to construct a subalgebra of  $A(\overline{\mathbb{D}})$  that has stable rank two: just take the restriction  $\mathbb{C}[z]|_{\overline{\mathbb{D}}}$  of the polynomials to  $\overline{\mathbb{D}}$ . In an oral communication Amol Sasane unveiled a first non-closed subalgebra of  $A(\overline{\mathbb{D}})$  with stable rank infinity: if  $\varphi$  is a conformal map of the disk  $\{|z| < 2\}$  onto the upper half plane  $H^+$ , then the algebra

$$A = \{f \circ \varphi|_{\overline{\mathbb{D}}} : f \in \text{AP}^+\}$$

of pull-backs of almost periodic functions that are analytic on  $H^+$  is isomorphic to  $\text{AP}^+$  and henceforth has stable rank infinity (see [5] and [7]).

It is the aim of this paper to prove, given  $n \in \mathbb{N} \cup \{\infty\}$ , the existence of *uniformly closed* subalgebras of  $A(\overline{\mathbb{D}})$  that have Bass stable rank  $n$ . The proof

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is based on embedding the polydisk algebras  $A(\overline{\mathbb{D}}^n)$  and  $A(\mathbf{D}^\infty)$  isomorphically into  $A(\overline{\mathbb{D}})$  (see below for the definitions). This will be done by using the Rudin-Carleson interpolation theorem for disk algebra functions and the topological fact (known under the name of the Alexandroff-Hausdorff theorem), that every compact metric space is the continuous image of the Cantor set (see for example [10]).

## 1. BACKGROUND

**Definition 1.1.** Let  $A$  be a commutative unital algebra (real or complex) with identity element denoted by 1.

- (1) An  $n$ -tuple  $(f_1, \dots, f_n) \in A^n$  is said to be *invertible* (or *unimodular*) if there exists  $(x_1, \dots, x_n) \in A^n$  such that the Bézout equation  $\sum_{j=1}^n x_j f_j = 1$  is satisfied. The set of all invertible  $n$ -tuples is denoted by  $U_n(A)$ . Note that  $U_1(A) = A^{-1}$ .  
An  $(n+1)$ -tuple  $(f_1, \dots, f_n, g) \in U_{n+1}(A)$  is called *reducible* if there exists  $(a_1, \dots, a_n) \in A^n$  such that  $(f_1 + a_1 g, \dots, f_n + a_n g) \in U_n(A)$ .
- (2) The *Bass stable rank* of  $A$ , denoted by  $\text{bsr } A$ , is the smallest integer  $n$  such that every element in  $U_{n+1}(A)$  is reducible. If no such  $n$  exists, then  $\text{bsr } A = \infty$ .

It is obvious that if  $A$  and  $B$  are two commutative unital algebras such that  $A$  is isomorphic to  $B$ , then  $\text{bsr } A = \text{bsr } B$ , because any isomorphism  $\iota$  between  $A$  and  $B$  induces a bijection between  $U_n(A)$  and  $U_n(B)$ . The following two observations stem from joint work with R. Rupp [8]. Here  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ .

**Proposition 1.2.** Let  $X$  be a topological space and  $B$  a subalgebra of  $C_b(X, \mathbb{K})$  with  $\mathbb{K} \subseteq B$ . Suppose that  $B$  has Royden's property  $(\alpha_0)$ ; that is

$(\alpha_0)$  For every  $f \in B$ : if  $\|1 - f\|_\infty < 1$ , then  $f \in B^{-1}$ .

Then  $\text{bsr } B \leq \text{bsr } \overline{B}^{\|\cdot\|_\infty}$ , where  $\overline{B}^{\|\cdot\|_\infty}$  is the uniform closure of  $B$ .

*Proof.* Let  $A := \overline{B}^{\|\cdot\|_\infty}$ . We show that  $U_n(B) = U_n(A) \cap B^n$ . Since  $U_n(B) \subseteq U_n(A) \cap B^n$ , it only remains to show the reverse inclusion. So let  $(b_1, \dots, b_n) \in U_n(A) \cap B^n$ . Then there is  $(a_1, \dots, a_n) \in A^n$  such that  $1 = \sum_{j=1}^n a_j b_j$ . Uniformly approximating  $a_j$  by elements  $x_j \in B$  yields that  $\|\sum_{j=1}^n x_j b_j - 1\|_\infty < 1/2$ . By assumption  $(\alpha_0)$ ,  $f := \sum_{j=1}^n x_j b_j \in B^{-1}$ . Hence  $(b_1, \dots, b_n) \in U_n(B)$ . It is now a standard observation that  $\text{bsr } B \leq \text{bsr } A$  (see [2] or [5]).  $\square$

**Corollary 1.3.** Let  $B$  be a subalgebra of the disk algebra  $A(\overline{\mathbb{D}})$  such that

- (1)  $B$  contains the polynomials (that is  $\mathbb{C}[z] \upharpoonright_{\overline{\mathbb{D}}} \subseteq B$ );
- (2) For every  $f \in B$ : if  $\|1 - f\|_\infty < 1$ , then  $f \in B^{-1}$ .

Then  $\text{bsr } B = 1$ .

*Proof.* By (1),  $B$  is uniformly dense in  $A(\overline{\mathbb{D}})$ . Because (2) is Royden's property  $(\alpha_0)$ , we may apply Proposition 1.2 to conclude that  $\text{bsr } B \leq \text{bsr } A(\overline{\mathbb{D}})$ . Since by the Jones-Marshall-Wolff theorem  $\text{bsr } A(\overline{\mathbb{D}}) = 1$  ([4]), we are done.  $\square$

## 2. AN EMBEDDING THEOREM

Recall that  $\overline{\mathbb{D}}^n$  is the closed polydisk and  $\mathbf{D}^\infty := \prod_{n \in \mathbb{N}} \overline{\mathbb{D}}$  the infinite polydisk. By Tychonov's theorem,  $\mathbf{D}^\infty$  is a compact metric space when endowed with the product topology. Moreover, each  $\overline{\mathbb{D}}^n$  and  $\mathbf{D}^\infty$  are separable. The *polydisk algebra*  $A(\overline{\mathbb{D}}^n)$  is the set of functions continuous on  $\overline{\mathbb{D}}^n$  and holomorphic on  $\mathbb{D}^n$ . In the same spirit, one defines the infinite polydisk algebra  $A(\mathbf{D}^\infty)$  as the smallest uniformly closed subalgebra of  $C(\mathbf{D}^\infty, \mathbb{C})$  containing all the coordinate functions  $z_1, z_2, \dots$ . Let  $\mathbb{C}[z_1, z_2, \dots]$  denote the set of polynomials

$$\sum_{\mathbf{j} \in \mathbb{N}^n} a_{\mathbf{j}} z_1^{j_1} \dots z_n^{j_n}, n \in \mathbb{N},$$

over  $\mathbb{C}$ , where  $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{N}^n$ . Hence

$$\mathbb{C}[z_1, z_2, \dots] |_{\mathbf{D}^\infty} \subseteq A(\mathbf{D}^\infty).$$

**Theorem 2.1.** *There are uniformly closed subalgebras  $A_n$  and  $A_\infty$  of  $A(\overline{\mathbb{D}})$  that are algebraically isomorphic to  $A(\overline{\mathbb{D}}^n)$ , respectively  $A(\mathbf{D}^\infty)$ .*

*Proof.* Let  $C \subseteq \mathbb{T}$  be the homeomorphic image of the usual ternary Cantor set on  $[0, 1]$  via the map  $e^{i\pi x}$ . By the Alexandroff-Hausdorff theorem, [10], there is a continuous surjective map

$$M_n = (\phi_1, \dots, \phi_n) : C \rightarrow \overline{\mathbb{D}}^n,$$

respectively

$$M_\infty = (\phi_1, \phi_2, \dots) : C \rightarrow \mathbf{D}^\infty.$$

Since  $C$  has one-dimensional Lebesgue-measure zero, the Rudin-Carleson interpolation Theorem [3, p. 58] implies that there are functions  $f_j \in A(\overline{\mathbb{D}})$  such that  $f_j|_C = \phi_j$  and  $\|f_j\| = 1$ . Define  $F_n : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}^n$  by

$$F_n(\xi) = (f_1(\xi), \dots, f_n(\xi)),$$

and  $F_\infty : \overline{\mathbb{D}} \rightarrow \mathbf{D}^\infty$  by

$$F_\infty(\xi) = (f_1(\xi), f_2(\xi), \dots).$$

By construction, the range of  $F_n$  on  $\overline{\mathbb{D}}$  is  $\overline{\mathbb{D}}^n$  and the range of  $F_\infty$  on  $\overline{\mathbb{D}}$  is  $\mathbf{D}^\infty$ . Moreover, since  $f_j(\mathbb{D}) \subseteq \mathbb{D}$ , the functions  $f \circ F_n$  and  $f \circ F_\infty$  are holomorphic on  $\mathbb{D}$  for any  $f \in A(\overline{\mathbb{D}}^n)$  respectively  $f \in A(\mathbf{D}^\infty)$ . Hence

$$\Psi_n : \begin{cases} A(\overline{\mathbb{D}}^n) & \rightarrow A(\overline{\mathbb{D}}) \\ f & \mapsto f \circ F_n \end{cases}$$

and

$$\Psi_\infty : \begin{cases} A(\mathbf{D}^\infty) & \rightarrow A(\overline{\mathbb{D}}) \\ f & \mapsto f \circ F_\infty \end{cases}$$

are isometric isomorphisms of  $A(\overline{\mathbb{D}}^n)$ , respectively  $A(\mathbf{D}^\infty)$ , onto a uniformly closed subalgebra of  $A(\overline{\mathbb{D}})$ .  $\square$

**Corollary 2.2.** *For every  $n \in \mathbb{N} \cup \{\infty\}$  there is a uniformly closed subalgebra  $A_n$  of  $A(\overline{\mathbb{D}})$  with  $\text{bsr } A_n = n$ .*

*Proof.* Let  $N \in \mathbb{N}$  be chosen so that  $\lfloor \frac{N}{2} \rfloor + 1 = n$ . By Theorem 2.1,  $A(\overline{\mathbb{D}}^N)$  is isomorphic to a uniformly closed subalgebra  $A_N$  of  $A(\overline{\mathbb{D}})$ . Hence

$$\text{bsr } A_N = \text{bsr } A(\overline{\mathbb{D}}^N) = \left\lfloor \frac{N}{2} \right\rfloor + 1 = n,$$

where the penultimate equality is due to Corach and Suárez [2]. Moreover, by [6],  $\text{bsr } A(\mathbf{D}^\infty) = \infty$ . Since by Theorem 2.1,  $A(\mathbf{D}^\infty)$  is isomorphic to a uniformly closed subalgebra  $A_\infty$  of  $A(\overline{\mathbb{D}})$ , we deduce that

$$\text{bsr } A_\infty = \text{bsr } A(\mathbf{D}^\infty) = \infty.$$

$\square$

### 3. THE TOPOLOGICAL STABLE RANK

Associated with the Bass stable rank is the notion of *topological stable rank* introduced by Rieffel [9].

**Definition 3.1.** Let  $A$  be a commutative unital complex Banach algebra. The *topological stable rank*,  $\text{tsr } A$ , of  $A$  is the least integer  $n$  for which  $U_n(A)$  is dense in  $A^n$ , or infinite if no such  $n$  exists.

It is straightforward to see (and well known) that  $\text{tsr } A(\overline{\mathbb{D}}) = 2$ . Corach and Suárez [1] showed that  $\text{tsr } A(\overline{\mathbb{D}}^n) = n + 1$  for  $n \in \mathbb{N}$ . Because  $\text{bsr } A \leq \text{tsr } A$  is always true,  $\text{tsr } A(\mathbf{D}^\infty) = \infty$ . Since the topological stable rank is invariant under isometric isomorphisms, we obtain from Corollary 2.1 the following theorem.

**Corollary 3.2.** *For every  $n \in \mathbb{N} \cup \{\infty\}$  there is a uniformly closed subalgebra  $A_n$  of  $A(\overline{\mathbb{D}})$  with  $\text{tsr } A_n = n + 1$ .*

*Proof.* Just take  $A_n := \Psi_n(A(\overline{\mathbb{D}}^n))$ , respectively  $A_\infty := \Psi_\infty(A(\mathbf{D}^\infty))$ .  $\square$

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## REFERENCES

- [1] G. Corach, F. D. Suárez, *Extension problems and stable rank in commutative Banach algebras*, Topology and its Applications 21 (1985), 1–8.
- [2] G. Corach, F. D. Suárez, *Dense morphisms in commutative Banach algebras*, Trans. Amer. Math. Soc. 304 (1987), 537–547.
- [3] T.W. Gamelin *Uniform algebras*, Chelsea, New York, 1984.
- [4] P.W. Jones, D. Marshall, T.H. Wolff, *Stable rank of the disc algebra*, Proc. Amer. Math. Soc. 96 (1986), 603–604.
- [5] K. Mikkola, A. Sasane, *Bass and topological stable ranks of complex and real algebras of measures, functions and sequences*, Complex Anal. Oper. Theory 4 (2010), 401–448.
- [6] R. Mortini, *An example of a subalgebra of  $H^\infty$  on the unit disk whose stable rank is not finite*, Studia Math. 103 (1992), 275–281.
- [7] R. Mortini, R. Rupp, *The Bass and topological stable ranks for algebras of almost periodic functions on the real line*, to appear in Trans. Amer. Math. Soc.
- [8] R. Mortini, R. Rupp, *An introduction to extension problems, Bézout equations and stable ranks in classical function algebras. A monograph accompanied by introductory chapters on point-set topology and function theory*, in preparation, ca. 1000 pages.
- [9] M. Rieffel, *Dimension and stable rank in the  $K$ -theory of  $C^*$ -algebras*, Proc. London Math. Soc. 46 (1983), 301–333.
- [10] I. Rosenholtz, *Another proof that any compact metric space is the continuous image of the Cantor set*, Amer. Math. Monthly 83 (1976), 646–647.

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